# A Corrected Algorithm for Computing the Theoretical Auto-Covariance Matrices of a Vector ARMA Model

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**Abstract:** The algorithm of Kohn and Ansley (1982) is reconsidered here, in order to correct several implementation errors concerning the construction of the linear equations that must be solved for computing the theoretical auto-covariance matrices of a vector ARMA model. This note presents a concise description of the corrected algorithm.

**Resumen:** En esta nota se corrigen algunos errores del algoritmo de Kohn y Ansley (1982), que tienen que ver con la construcción de un sistema de ecuaciones lineales para calcular las matrices de auto-covarianzas teóricas de un modelo ARMA multivariante.

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## 1. INTRODUCTION

Consider a stationary vector time-series process ..., $\mathbf{W}_{-1}$, $\mathbf{W}_0$, $\mathbf{W}_1$, ... of dimension $m \geq 2$ (with $\mathbf{W}_t = [W_{t1}, \ldots, W_{tm}]^{\mathrm{T}}$) following the vector ARMA$(p, q)$ model

$$\tilde{\mathbf{W}}_t = \mathbf{\Phi}_1 \tilde{\mathbf{W}}_{t-1} + \mathbf{\Phi}_2 \tilde{\mathbf{W}}_{t-2} + \cdots + \mathbf{\Phi}_p \tilde{\mathbf{W}}_{t-p} + \mathbf{A}_t - \mathbf{\Theta}_1 \mathbf{A}_{t-1} - \mathbf{\Theta}_2 \mathbf{A}_{t-2} - \cdots - \mathbf{\Theta}_q \mathbf{A}_{t-q}, \tag{1}$$

where $\tilde{\mathbf{W}}_i = \mathbf{W}_i - \mathrm{E}[\mathbf{W}_i]$, $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_p$ and $\mathbf{\Theta}_1, \ldots, \mathbf{\Theta}_q$ are $m \times m$ parameter matrices, and ..., $\mathbf{A}_{-1}$, $\mathbf{A}_0$, $\mathbf{A}_1$, ... is a sequence of IID$(\mathbf{0}, \mathbf{\Sigma})$ $m \times 1$ random vectors with $\mathbf{\Sigma}$ ($m \times m$) symmetric and positive definite.

Let the theoretical "auto-covariance" matrix ($m \times m$) of order $k \in \mathbb{Z}$ be defined as

$$\mathbf{\Gamma}_k = \mathrm{E}[\tilde{\mathbf{W}}_t \tilde{\mathbf{W}}_{t+k}^{\mathrm{T}}] = \mathrm{E}[\tilde{\mathbf{W}}_{t-k} \tilde{\mathbf{W}}_t^{\mathrm{T}}].$$

Then, transposing and premultiplying (1) by $\tilde{\mathbf{W}}_{t-k}$, and taking expectations, it is easily verified that

$$\mathbf{\Gamma}_k = \sum_{i=1}^{p} \mathbf{\Gamma}_{k-i} \mathbf{\Phi}_i^{\mathrm{T}} + \mathbf{\Lambda}_k - \sum_{j=1}^{q} \mathbf{\Lambda}_{k-j} \mathbf{\Theta}_j^{\mathrm{T}}, \tag{2}$$

where $\mathbf{\Lambda}_i = \mathrm{E}[\tilde{\mathbf{W}}_t \mathbf{A}_{t+i}^{\mathrm{T}}] = \mathrm{E}[\tilde{\mathbf{W}}_{t-i} \mathbf{A}_t^{\mathrm{T}}]$ is the theoretical "cross-covariance" matrix ($m \times m$) of order $k$.

The theoretical auto-covariance and cross-covariance matrices of various orders are needed, for example, to compute the objective function during exact maximum likelihood estimation of (1) (see, among others, Hall and Nicholls 1980, Shea 1989, and Mauricio 1995). Other applications include the generation of independent realizations from a vector ARMA model (see, for example, Shea 1988), which is particularly useful in simulation studies. Thus, as Kohn and Ansley (1982) point out, it is important to have a fast method of obtaining those matrices, since they are recalculated many times during both estimation and simulation runs.

The method of Kohn and Ansley (1982) provides a fast means of computing the auto-covariance matrices. In particular, it is faster than the methods of Hall and Nicholls (1980) and Ansley (1980). However, the implementation in the algorithmic section of Kohn and Ansley (1982, pp. 278-279) is incorrect. In particular, the guidelines on the construction of the linear equations that must be solved to compute $\mathbf{\Gamma}_0$ through $\mathbf{\Gamma}_{p-1}$ are wrong. A corrected version of that algorithm, based on the theoretical background of Section 2, is presented in Section 3. The construction of both the matrix and the right-hand-side vector of the system of linear equations for the first $p$ auto-covariance matrices is described in detail. Some further comments in Section 4 complete this note.

## 2. THEORETICAL BACKGROUND

In order to motivate the algorithm of Section 3, a concise summary of the method proposed by Kohn and Ansley (1982), using the notation introduced in Section 1, is given now. In the first place, postmultiplying (1) by $\mathbf{A}_{t+i}^{\mathrm{T}}$, taking expectations, and noting that $\mathrm{E}[\tilde{\mathbf{W}}_{t-i}\mathbf{A}_t^{\mathrm{T}}] = \mathrm{E}[\mathbf{A}_t\mathbf{A}_{t+i}^{\mathrm{T}}] = \mathbf{0}$ for $i \geq 1$, it can be verified that

$$\mathbf{\Lambda}_{-i} = -\mathbf{\Theta}_i\mathbf{\Sigma} + \sum_{j=1}^{p} \mathbf{\Phi}_j\mathbf{\Lambda}_{j-i} \quad \text{for } i \geq 1, \tag{3}$$

where $\mathbf{\Theta}_i = \mathbf{0}$ for $i > q$, $\mathbf{\Lambda}_0 = \mathbf{\Sigma}$, and $\mathbf{\Lambda}_k = \mathbf{0}$ for $k > 0$. Then, writing out (2) for $k = 0$ and for $k \geq 1$,

$$\mathbf{\Gamma}_0 = \sum_{i=1}^{p} \mathbf{\Gamma}_{-i}\mathbf{\Phi}_i^{\mathrm{T}} + \mathbf{\Lambda}_0 - \sum_{j=1}^{q} \mathbf{\Lambda}_{-j}\mathbf{\Theta}_j^{\mathrm{T}}, \tag{4}$$

$$\mathbf{\Gamma}_k = \mathbf{C}_k + \sum_{j=1}^{p} \mathbf{\Gamma}_{k-j}\mathbf{\Phi}_j^{\mathrm{T}} \quad \text{for } k \geq 1, \tag{5}$$

where

$$\mathbf{C}_k = -\sum_{j=k}^{q} \mathbf{\Lambda}_{k-j}\mathbf{\Theta}_j^{\mathrm{T}} \quad \text{for } k \geq 1. \tag{6}$$

Now, noting that $\mathbf{\Gamma}_{-i} = \mathrm{E}[\tilde{\mathbf{W}}_t\tilde{\mathbf{W}}_{t-i}^{\mathrm{T}}] = \mathrm{E}[\tilde{\mathbf{W}}_{t+i}\tilde{\mathbf{W}}_t^{\mathrm{T}}] = \mathbf{\Gamma}_i^{\mathrm{T}}$, and putting (5) with $k$ replaced by $i$ into (4), it turns out that

$$\mathbf{\Gamma}_0 = \mathbf{C}_0 + \sum_{j=1}^{p}\sum_{i=1}^{p} \mathbf{\Phi}_i\mathbf{\Gamma}_{i-j}\mathbf{\Phi}_j^{\mathrm{T}}, \tag{7}$$

where

$$\mathbf{C}_0 = \mathbf{\Sigma} - (\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \sum_{j=1}^{q} \mathbf{\Theta}_j\mathbf{\Sigma}\mathbf{\Theta}_j^{\mathrm{T}} \tag{8}$$

and

$$\mathbf{B} = \sum_{i=1}^{p}\sum_{j=i}^{q} \mathbf{\Phi}_i\mathbf{\Lambda}_{i-j}\mathbf{\Theta}_j^{\mathrm{T}}. \tag{9}$$

Finally, since $\mathbf{\Gamma}_{-k} = \mathbf{\Gamma}_k^{\mathrm{T}}$, it can be verified that when (5) and (6) are written for $k$ = 1, 2, ..., $p-1$, only $\mathbf{\Gamma}_0$ through $\mathbf{\Gamma}_{p-1}$ and $\mathbf{\Lambda}_0$ through $\mathbf{\Lambda}_{-q+1}$ appear in equations (5) through (9). Thus, given $\mathbf{\Lambda}_0$ through $\mathbf{\Lambda}_{-q+1}$, the following system of linear equations may be specified in order to calculate $\mathbf{\Gamma}_0$

through $\mathbf{\Gamma}_{p-1}$:

$$\mathbf{\Gamma}_0 - \sum_{i=1}^{p} \mathbf{\Phi}_i \mathbf{\Gamma}_0 \mathbf{\Phi}_i^{\mathrm{T}} - \sum_{i=1}^{p-1} \sum_{j=1}^{p-i} \left( \mathbf{\Phi}_{i+j} \mathbf{\Gamma}_i \mathbf{\Phi}_j^{\mathrm{T}} + \mathbf{\Phi}_j \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Phi}_{i+j}^{\mathrm{T}} \right) = \mathbf{C}_0 , \tag{10}$$

$$\mathbf{\Gamma}_k - \sum_{i=1}^{k-1} \mathbf{\Gamma}_i \mathbf{\Phi}_{k-i}^{\mathrm{T}} - \sum_{i=0}^{p-k} \mathbf{\Gamma}_i^{\mathrm{T}} \mathbf{\Phi}_{k+i}^{\mathrm{T}} = \mathbf{C}_k \quad \text{for } k = 1, \ldots, p-1 . \tag{11}$$

Together, equations (10) and (11) form a system of $pm^2$ linear equations with $pm^2$ unknowns (the $m^2$ elements of each of the $p$ matrices $\mathbf{\Gamma}_0$, $\mathbf{\Gamma}_1$, ..., $\mathbf{\Gamma}_{p-1}$. But since $\mathbf{\Gamma}_0$ is symmetric, it contains only $m(m+1)/2$ distinct elements. Thus, letting $\tilde{\mathbf{\Gamma}}_0$ and $\tilde{\mathbf{C}}_0$ be the diagonals and upper triangles of $\mathbf{\Gamma}_0$ and $\mathbf{C}_0$, respectively, the $m(m+1)/2 + m^2(p-1)$ unknowns in (10) and (11) are the (unique) solution $\mathbf{x}$ to the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x} = \mathrm{vec}[\tilde{\mathbf{\Gamma}}_0, \mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_{p-1}]$, $\mathbf{b} = \mathrm{vec}[\tilde{\mathbf{C}}_0, \mathbf{C}_1, \ldots, \mathbf{C}_{p-1}]$, and $\mathbf{A}$ is the matrix of coefficients, which (together with $\mathbf{b}$) is constructed as described in the next section.

## 3. THE CORRECTED ALGORITHM

In order to concentrate on the more relevant aspects of the algorithm (the construction of $\mathbf{A}$ and $\mathbf{b}$), it is assumed that $\mathbf{\Lambda}_0$ and $\mathbf{\Lambda}_{-1}$ through $\mathbf{\Lambda}_{-q+1}$, as well as $\tilde{\mathbf{C}}_0$, have already been computed using (3), (8) and (9). Now, it is shown how to construct $\mathbf{A}$ and $\mathbf{b}$ row by row. First, initialize $\mathbf{A}$ and $\mathbf{b}$ to zero; then, execute the following two steps:

**Step 1**. Compute the first $m(m+1)/2$ rows:

FOR $j = 1$ TO $m$

  FOR $i = 1$ TO $j$

    $row = j \times (j-1)/2 + i$

**Step 1.1**. Compute the first $m(m+1)/2$ columns within *row*:

FOR $l = 1$ TO $m$

  FOR $k = 1$ TO $l$

    $col = l \times (l-1)/2 + k$

    IF $k = l$

$$\mathbf{A}(row, col) = -\sum_{r=1}^{p} \mathbf{\Phi}_r(i,k) \times \mathbf{\Phi}_r(j,l)$$

ELSE

$$\mathbf{A}(row, col) = -\sum_{r=1}^{p} \left[ \mathbf{\Phi}_r(i,k) \times \mathbf{\Phi}_r(j,l) + \mathbf{\Phi}_r(i,l) \times \mathbf{\Phi}_r(j,k) \right]$$

**Step 1.2**. Compute the remaining $m^2(p-1)$ columns within *row*:

FOR $s = 1$ TO $p-1$

FOR $l = 1$ TO $m$

FOR $k = 1$ TO $m$

$col = m \times (m+1)/2 + m^2 \times (s-1) + m \times (l-1) + k$

$$\mathbf{A}(row, col) = -\sum_{r=1}^{p-s} \left[ \mathbf{\Phi}_{r+s}(i,k) \times \mathbf{\Phi}_r(j,l) + \mathbf{\Phi}_{r+s}(j,k) \times \mathbf{\Phi}_r(i,l) \right]$$

**Step 1.3**. Set up diagonal of **A** and right-hand-side **b**:

$\mathbf{A}(row, row) = 1 + \mathbf{A}(row, row)$

$\mathbf{b}(row) = \tilde{\mathbf{C}}_0(i,j)$

**Step 2**. Compute the remaining $m^2(p-1)$ rows:

FOR $s = 1$ TO $p-1$

FOR $i = 1$ TO $m$

FOR $j = 1$ TO $m$

$row = m \times (m+1)/2 + m^2 \times (s-1) + m \times (i-1) + j$

**Step 2.1**. Compute the first $m(m+1)/2$ columns within *row*:

FOR $l = 1$ TO $m$

IF $l \le j$

$col = j \times (j-1)/2 + l$

ELSE

$col = l \times (l-1)/2 + j$

$\mathbf{A}(row, col) = -\mathbf{\Phi}_s(i,l)$

**Step 2.2**. Compute the remaining $m^2(p-1)$ columns within *row*:

FOR $r = 1$ TO $p-1$

FOR $l = 1$ TO $m$

$col = m\times(m+1)/2 + m^2\times(r-1) + m\times(j-1) + l$

IF $r + s \leq p$

$\mathbf{A}(row, col) = -\mathbf{\Phi}_{r+s}(i, l)$

IF $s > r$

$col = m\times(m+1)/2 + m^2\times(r-1) + m\times(l-1) + j$

$\mathbf{A}(row, col) = \mathbf{A}(row, col) - \mathbf{\Phi}_{s-r}(i, l)$

**Step 2.3**. Set up diagonal of **A** and right-hand-side **b**:

$\mathbf{A}(row, row) = 1 + \mathbf{A}(row, row)$

FOR $h = s$ TO $q$

$$\mathbf{b}(row) = \mathbf{b}(row) - \sum_{k=1}^{m} \mathbf{\Lambda}_{s-h}(j, k) \times \mathbf{\Theta}_h(i, k)$$

END.

Once **A** and **b** are constructed, the solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ may be obtained using standard numerical linear algebra routines (see, for example, Moler 1972, and Press et al. 1992). Then, the elements of the theoretical auto-covariance matrices can be extracted from **x** as follows:

$$\mathbf{\Gamma}_0(i, j) = \mathbf{x}_{j(j-1)/2+i} \quad (i = 1, \ldots, m; j = i, \ldots, m),$$
$$\mathbf{\Gamma}_k(i, j) = \mathbf{x}_{m(m+1)/2+m^2(k-1)+m(j-1)+i} \quad (i, j = 1, \ldots, m; k = 1, \ldots, p-1).$$

For $k \geq p$, the auto-covariance matrices can be computed recursively using (5), (6) and (3).

## 4. CONCLUDING REMARKS

Note that, except for the first $m(m+1)/2$ coefficients in each of the first $m(m+1)/2$ rows of **A** (Step 1.1 above), the implementation presented here differs from that of Kohn and Ansley (1982), which is incorrect. To check this, computer programs were written by the author, following the implementation in

Kohn and Ansley (1982) and the one described in Section 3. The theoretical auto-covariance matrices, calculated through the implementation in Kohn and Ansley (1982) for a wide range of vector ARMA models, always showed substantial differences with the results obtained through the implementation described here. Furthermore, the latter results always agreed with those obtained through standard procedures, such as that of Hall and Nicholls (1980) as implemented in Shea (1989).

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